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Completely Analytical Interactions: Constructive Description

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An interaction U is called a completely analytical (CA) interaction, if it satisfies one of 12 given conditions formulated in terms of analyticity properties of the partition functions $Z_V(U)$, or correlation decay, or truncated correlation bounds, or asymptotic behavior of $\ln Z_V(U)$, $V \to \infty$. The 12 conditions are presented, together with part of the proof of their equivalence. The main result of the paper is that each condition is constructive in the following sense: instead of checking it in all finite volumes $V \subset \mathbb{Z}^\nu$, it is enough to consider only (a finite amount of) volumes with restricted size. In particular, the partition functions $Z_V(U + \tilde{U})$ for the complex perturbations $U + \tilde{U}$ of U do not vanish for all $V \subset \mathbb{Z}^\nu$ and all $\tilde{U}$ with $\|\tilde{U}\| < \varepsilon$, provided this is true only for V with diam $V \leqslant C(\varepsilon)$ and $\|\tilde{U}\| < \varepsilon'$ (but with $\varepsilon < \varepsilon'$).

KEY WORDS: Analyticity; correlation decay; Gibbs states; uniqueness; surgery method.

1. INTRODUCTION

"All happy families are alike, each unhappy family is unhappy in its own fashion." This observation from the opening lines of Leo Tolstoy *Anna Karenin* can well serve as an epigraph to the family of papers that includes the present one.[1,2] The main goal of these paper is to demonstrate that, contrary to the richness of the behavior exhibited by Gibbs fields at low temperatures, their properties outside the phase transition region are quite uniform. In Ref. 1 we introduced nine properties (increased to 12 in Ref. 2) of a very natural kind, which are formulated in terms of bounds on the partition function in the complex region, or on semi-invariants, or on correlation decay, and it turned out *a posteriori* that each of them defines

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the same class of interactions (however, under an additional restriction; see Section 2). This class is called the class of completely analytical (CA) potentials ("happy" ones).

Each of the 12 properties of the CA potentials has the following structure: one asks for a certain bound on the partition function or another quantity in a finite volume V to hold for each boundary condition. What is important here is that the bound in question has to be uniform in V. Even in the case of finite-range interactions and finite spins, to which we shall restrict ourselves for simplicity, these conditions are not constructive, in the sense that one has to perform an infinite number of checks to verify them. The main goal of the present paper is to show the existence of constructive criteria of CA interactions. They are of the same type as the nonconstructive ones, with the main difference that the corresponding bounds have to be checked only for the volumes V inside some cube, the size of which is explicitly estimated by certain functions of the constants, which enter the above bounds. These criteria are effective in the following sense: one can write down a computer program such that if a CA interaction is substituted into it, then it will check its complete analyticity in finite time (depending, of course, on the interaction). At the same time, for a nonanalytic interaction the program will never stop. So, using the language of algorithm theory, the set of CA interactions is enumerable (if one considers only the interactions with rational values), though not necessarily calculable. This enables one to prove the CA for a given interaction by the help of a computer. For another problem this possibility was discussed earlier.[3,4]

In Section 2 we repeat the definitions of CA from Refs. 1 and 2. We omit the proofs and the discussions of Ref. 1, but for the benefit of the reader we reproduce those of Ref. 2, which are contained in Section 3. In Section 4 the constructive variants of CA conditions are given, while Sections 5 and 6 contain the proofs, which are technically more involved than the proof of the equivalence of different nonconstructive criteria.

Throughout this paper the following notations will be used: $\mathbb{Z}^\nu$ is the ν-dimensional lattice with points $t=(t^1,\dots, t^\nu)$; $\operatorname{dist}(s,t)=\max_{i=1,\dots,\nu}|s^i-t^i|$, $s, t\in\mathbb{Z}^\nu$; $\mathscr{S}$ is a finite single spin set; $V, W, \Lambda,\dots\subset\mathbb{Z}^\nu$ are finite volumes; $|X|$ is the number of points in a finite set X; $V^c=\mathbb{Z}^\nu\setminus V$; $D_n=\{t\in\mathbb{Z}^\nu\colon -n\leqslant t^i\leqslant n,\ i=1,\dots,\nu\}$ is a cube centered at the origin; Ω_V is the set of configurations on V: $\Omega_V=\{\sigma_V: V\to\mathscr{S}\}$; $\Omega=\Omega_{Z^\nu}$; $\sigma, \bar\sigma\in\Omega$ are configurations; $\sigma_V=\sigma|_V$ is the restriction of $\sigma\in\Omega$ or $\sigma\in\Omega_W$, $V\subset W$, on V; $\sigma_{V_1}\cup\sigma_{V_2}\in\Omega_{V_1\cup V_2}$ is such that $(\sigma_{V_1}\cup\sigma_{V_2})_{V_i}=\sigma_{V_i}$ for $V_1, V_2\subset\mathbb{Z}^\nu$, $V_1\cap V_2=\varnothing$, $\sigma_{V_i}\in\Omega_{V_i}$, $i=1,2$; and $\partial V=\partial_r V=\{t\in V^c\colon \operatorname{dist}(t,V)\leqslant r\}$ for $r>0$.

2. THE CRITERIA FOR COMPLETE ANALYTICITY

Let $\mathscr{A} = \{A_i \subset \mathbb{Z}^\nu,\ i = 1,..., k\}$ be a finite collection of finite subsets of $\mathbb{Z}^\nu$ and $\Delta = \Delta(\mathscr{A}) = \{A \subset \mathbb{Z}^\nu: A = A_i + t$ for some $i = i(A) = 1,..., k$, $t = t(A) \in \mathbb{Z}^\nu\}$. We denote by $\mathfrak{U}_\Delta$ and $\mathfrak{U}_\Delta^{\mathbb{C}}$ the real and the complex Banach spaces of translation-invariant interactions with support in Δ. An interaction $U \in \mathfrak{U}_\Delta$ ($\mathfrak{U}_\Delta^{\mathbb{C}}$) is thus a family $U = \{U_A(\sigma) \equiv U_A(\sigma_A),\ A \subset \mathbb{Z}^\nu,\ |A| < \infty,\ \sigma \in \Omega\}$ such that

$$U_A(\sigma) = U_{A+t}(\sigma_{+t}) \qquad \text{where} \quad (\sigma_{+t})_s = \sigma_{s-t} \tag{2.1}$$

$$U_A \equiv 0 \qquad \text{unless} \quad A \in \Delta(\mathscr{A}) \tag{2.2}$$

The norm of U is given by

$$\|U\| = \sup_{A,\sigma} |U_A(\sigma)| \tag{2.3}$$

The radius of interaction U is the number

$$r = r(U) = \max_{A:U_A \not\equiv 0} \operatorname{diam} A$$

If $\Delta = \{A \subset \mathbb{Z}^\nu: \operatorname{diam} A \leqslant r\}$, then the corresponding spaces $\mathfrak{U}_\Delta$ and $\mathfrak{U}_\Delta^{\mathbb{C}}$ also will be denoted by $\mathfrak{U}_r$ and $\mathfrak{U}_r^{\mathbb{C}}$. Throughout most of this section the families $\mathscr{A}$ and Δ will be fixed and so the corresponding index will be omitted, and we shall speak of the spaces $\mathfrak{U}$ and $\mathfrak{U}^{\mathbb{C}}$ with $r = r(\mathfrak{U})$ being the maximum radius of interactions $U \in \mathfrak{U}$.

For any set $\mathfrak{A} \in \mathfrak{U}$ we define its main component $M(\mathfrak{A})$ to be the maximal open connected subset of $\mathfrak{A}$ that contains the zero interaction $U^0 = \{U_A^0 \equiv 0\}$.

We shall denote by $\mathfrak{A}_\alpha$ the set of interactions satisfying condition α, where α denotes one of the 12 conditions to be formulated below (α = Ia–Ic, IIa–IIc, IIIa–IIId, IVa, IVb).

We now present the main result concerning the equivalence of different CA conditions.

Theorem 2.1. The main components $M(\mathfrak{A}_\alpha)$ coincide (α = Ia–IVb).

This common main component is called the set of completely analytical potentials.

We now start the formulation of our 12 conditions.

Let $\tilde{\mathfrak{U}}$ ($\tilde{\mathfrak{U}}^{\mathbb{C}}$) be the set of all real (complex) interactions that satisfy (2.2), but not necessarily (2.1), with the norm (2.3). Of course, $\mathfrak{U} \subset \tilde{\mathfrak{U}}$ and $\mathfrak{U}^{\mathbb{C}} \subset \tilde{\mathfrak{U}}^{\mathbb{C}}$. For any $U \in \tilde{\mathfrak{U}}^{\mathbb{C}}$, $V \subset \mathbb{Z}^\nu$ finite, and boundary condition $\bar{\sigma} \in \Omega$ let

$$Z_V(U \mid \bar{\sigma}) = \sum_{\sigma_V \in \Omega_V} \exp[-H_V(\sigma_V \mid \bar{\sigma})] \tag{2.4}$$

where

$$H_V(\sigma_V \mid \bar\sigma) = \sum_{A: A \cap V \neq \varnothing} U_A(\sigma_V \cup \bar\sigma_{V^c}) \tag{2.5}$$

Condition Ia. $U \in \mathfrak{A}_{\mathrm{Ia}}$ iff there exists $\varepsilon > 0$ such that for all V, $\bar\sigma$ the (analytic) functions $Z_V(\tilde U \mid \bar\sigma)$ are nonvanishing, provided

$$\tilde U \in O_\varepsilon^T(U) = \{\tilde U \in \mathfrak{U}^{\mathbb{C}} : \|U - \tilde U\| < \varepsilon\} \tag{2.6}$$

(i.e., for all small enough translation-invariant complex perturbations of U).

Condition Ib. $U \in \mathfrak{A}_{\mathrm{Ib}}$ iff there exist $C < \infty$ and $\varepsilon > 0$ such that for all V, $\bar\sigma$ the (analytic) functions $Z_V(\tilde U \mid \bar\sigma)$ are nonvanishing, provided

$$\tilde U \in O_\varepsilon(U) = \{\tilde U \in \tilde{\mathfrak{U}}^{\mathbb{C}} : \|\tilde U - U\| < \varepsilon\} \tag{2.7}$$

and, moreover,

$$\begin{aligned} &|\ln[Z_V(\tilde U_1 \mid \bar\sigma)/Z_V(\tilde U_2 \mid \bar\sigma)]| \\ &\qquad \leqslant C\,|(V \cup \partial V) \cap \operatorname{supp}(\tilde U_1 - \tilde U_2)| \end{aligned} \tag{2.8}$$

for all $\tilde U_1, \tilde U_2 \in O_\varepsilon(U)$, where, for $\Phi \in \tilde{\mathfrak{U}}^{\mathbb{C}}$,

$$\operatorname{supp} \Phi = \bigcup_{A: \Phi_A \not\equiv 0} A$$

Remark 1. The function $Z_V(\tilde U \mid \bar\sigma)$ depends only on those $\tilde U_A(\sigma_A)$ for which $A \cap V \neq \varnothing$. By being holomorphic we mean the usual property of functions of several complex variables.

Remark 2. The function $\ln Z_V(\tilde U \mid \bar\sigma)$ is a uniquely defined holomorphic function, which coincides with the usual (real) logarithm for $\tilde U$ real. Its analytic continuation to $O_\varepsilon(U)$ is possible and unique because the latter set is contractivle and $Z \mid O_\varepsilon(U) \neq 0$.

Condition Ic. $U \in \mathfrak{A}_{\mathrm{Ic}}$ iff there exists $\varepsilon > 0$ such that for all V and $\bar\sigma$, $Z_V(\tilde U \mid \bar\sigma)$ is nonvanishing for $\tilde U \in O_\varepsilon(U)$, and for any complex function φ on Ω, which is $\mathscr{B}_W$-measurable for some $W \subset V$,

$$|\langle \varphi \rangle_{V,\bar\sigma}^{\tilde U}| \leqslant \tilde C\,\|\varphi\| \tag{2.9}$$

with $\|\varphi\| = \sup_\sigma |\varphi(\sigma)|$, where $\tilde C = \tilde C(|W|, U, r, \nu, \varepsilon, |\mathscr{S}|)$, uniformly in V. Here

$$\langle \varphi \rangle_{V,\bar\sigma}^{\tilde U} = \int_{\Omega_V} \varphi(\sigma_V \cup \bar\sigma_{V^c})\, Q_V^{\tilde U}(\sigma_V \mid \bar\sigma)\, d\sigma_V$$

is the expectation with respect to the conditional Gibbs measure in V with complex interaction, with

$$Q_V^{\tilde{U}}(\sigma_V \mid \bar{\sigma}) = \exp[-\tilde{H}_V(\sigma_V \mid \bar{\sigma})]/Z_V(\tilde{U} \mid \bar{\sigma}) \tag{2.10}$$

where $\tilde{H}_{\cdot}(\cdot \mid \cdot)$ is defined by (2.5) with $\tilde{U}$ instead of U. This measure is well-defined because the partition function does not vanish. In case $\tilde{U}$ is real, the bound (2.9) holds trivially.

We recall now the definition of semi-invariants. Let $\xi_1,..., \xi_m$ be random variables with values in $\mathscr{S}$ and with the joint probability distribution $q(x_1,..., x_m)$, $x_i \in \mathscr{S}$. The semi-invariant of order $(k_1,..., k_m)$, where $k_i > 0$, is the number

$$\begin{aligned} &\langle \xi_1^{;k_1}, \xi_2^{;k_2},..., \xi_m^{;k_m} \rangle_q \\ &\quad = \frac{\partial^{k_1 + \cdots + k_m}}{\partial^{k_1} z_1 \cdots \partial^{k_m} z_m} \ln \varphi(z_1,..., z_m) \mid_{z_1 = z_2 = \cdots = 0} \end{aligned} \tag{2.11}$$

where

$$\varphi(z_1,..., z_m) = \sum_{x_1,...,x_m \in \mathscr{S}} \exp(z_1 x_1 + \cdots + z_m x_m)\, q(x_1,..., x_m) \tag{2.12}$$

is the generating function, and $z_i \in \mathbb{C}$.

Now let $\psi_1(\sigma_{A_1}),..., \psi_m(\sigma_{A_m})$ be real functions, where $A_i \subset V$ are (not necessarily distinct) subsets; then, for $\tilde{U} \in \tilde{\mathfrak{U}}$,

$$\begin{aligned} &\langle \psi_1^{;k_1},..., \psi_m^{;k_m} \rangle_{Q_V^{\tilde{U}}(\cdot \mid \bar{\sigma})} \\ &\quad \equiv \langle \psi_1^{;k_1},..., \psi_m^{;k_m} \mid \tilde{U}, V, \bar{\sigma} \rangle \\ &\quad = \frac{\partial^{k_1 + \cdots + k_m}}{\partial^{k_1} z_1 \cdots \partial^{k_m} z_m} [\ln Z_V(\hat{U}(z_1,..., z_m) \mid \bar{\sigma})] \mid_{z_1 = \cdots = z_m = 0} \end{aligned} \tag{2.13}$$

where

$$(\hat{U}(z_1,..., z_m))_A = \tilde{U}_A + \sum_{i: A_i = A} z_i \psi_i \tag{2.14}$$

Condition IIa. $U \in \mathfrak{A}_{\text{IIa}}$ iff for some constants $C < \infty$ and $\varepsilon > 0$ and for all $V, \bar{\sigma}, m, \psi_1,..., \psi_m, k_1,..., k_m$ with $|\psi_i| \leqslant 1$ and $A_i \in \Delta$, the function $\langle \psi_1^{;k_1},..., \psi_m^{;k_m} \mid \tilde{U}, V, \bar{\sigma} \rangle$ defined by (2.13) for real $\tilde{U} \in \tilde{\mathfrak{U}}$ can be extended to a holomorphic function on $O_\varepsilon(U)$ with the following bound:

$$|\langle \psi_1^{;k_1},..., \psi_m^{;k_m} \mid \tilde{U}, V, \bar{\sigma} \rangle| \leqslant k_1! \cdots k_m!\, C^{k_1 + \cdots + k_m} \tag{2.15}$$

Condition IIb. $U \in \mathfrak{A}_{\mathrm{IIb}}$ iff there exists a constant $C < \infty$ such that for all $V, \bar{\sigma}, m, \psi_1, ..., \psi_m, k_1, ..., k_m$ with $|\psi_i| \leqslant 1$ and $A_i \in \Delta$

$$|\langle \psi_1^{;k_1}, ..., \psi_m^{;k_m} \mid U, V, \bar{\sigma} \rangle| \leqslant k_1! \cdots k_m!\, C^{k_1 + \cdots + k_m} \sum_{\Gamma \in G(A_1, ..., A_m)} \prod_{\gamma \in \mathscr{E}(\Gamma)} \varphi(|\gamma|) \tag{2.16}$$

where $G(A_1, ..., A_m)$ is the set of all trees Γ with m vertices identified with the sets $A_1, ..., A_m$, $\mathscr{E}(\Gamma)$ is the set of all bonds $\gamma = (A_{i_\gamma}, A_{j_\gamma})$ of Γ, $|\gamma| = \mathrm{dist}(A_{i_\gamma}, A_{j_\gamma})$, and finally $\varphi(d) > 0$ is a decreasing function on $\mathbb{Z}_+$ with

$$\sum_{t \in \mathbb{Z}^\nu} \varphi(|t|)\, |t|^{\nu - 1} < \infty \tag{2.17}$$

Condition IIc. $U \in \mathfrak{A}_{\mathrm{IIc}}$ iff for some constants $C < \infty$ and $\delta > 0$ and for all $V, \bar{\sigma}, m, \psi_1, ..., \psi_m, k_1, ..., k_m$ with $|\psi_i| \leqslant 1$, and $A_i \in \Delta$

$$|\langle \psi_1^{;k_1}, ..., \psi_m^{;k_m} \mid U, V, \bar{\sigma} \rangle| \leqslant k_1! \cdots k_m!\, C^{k_1 + \cdots + k_m} \exp[-\delta\, d(A_1, ..., A_m)] \tag{2.18}$$

where

$$d(A_1, ..., A_m) = \min_{B : B \cup (A_1 \cup \cdots \cup A_m) \text{ is connected}} |B|$$

and the connectedness is meant in the sense of the graph $\mathbb{Z}^\nu$ with edges joining nearest neighbors.

For $\Lambda \subset V$ we define

$$Q_{V,\Lambda}^U(B \mid \bar{\sigma}) = \sum_{\sigma_V \in \Omega_V : \sigma_\Lambda \in B} Q_V^U(\sigma_V \mid \bar{\sigma}), \qquad B \subset \Omega_\Lambda \tag{2.19}$$

Condition IIIa. $U \in \mathfrak{A}_{\mathrm{IIIa}}$ iff for some constants $\delta < 1$ and $\rho > 0$ and for all finite $V \subset \mathbb{Z}^\nu$, $t \in \partial V$, $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ with $\bar{\sigma}_s^1 = \bar{\sigma}_s^2$ for $s \neq t$,[2]

$$\mathrm{Var}(Q_{V,B(t,\rho,V)}^U(\cdot \mid \bar{\sigma}^1), Q_{V,B(t,\rho,V)}^U(\cdot \mid \bar{\sigma}^2)) < \tfrac{1}{2}\delta\, |B(t, \rho, V)|^{-1} \tag{2.20}$$

where

$$B(t, \rho, V) = \{s \in V : \rho < |s - t| \leqslant \rho + r\}, \qquad r = r(U) \tag{2.21}$$

We denote by Var the variation distance: if Q_1, Q_2 are probability measures on a finite set X, then

$$\mathrm{Var}(Q_1, Q_2) = \tfrac{1}{2} \sum_{x \in X} |Q_1(x) - Q_2(x)|$$

[2] In Ref. 1, the factor 1/2 was missed.

Condition IIIb. $U \in \mathfrak{A}_{\text{IIIb}}$ iff for some decreasing function $\varphi(d)$ with

$$\lim_{d \to \infty} \varphi(d)\, d^{2(\nu - 1)} = 0 \tag{2.22}$$

for the same V, t, $\bar{\sigma}^1$, $\bar{\sigma}^2$ as in IIIa, and for any $\Lambda \subset V$,

$$\operatorname{Var}(Q^U_{V,\Lambda}(\cdot \mid \bar{\sigma}^1), Q^U_{V,\Lambda}(\cdot \mid \bar{\sigma}^2)) \leqslant \sum_{s \in \Lambda} \varphi(|s - t|) \tag{2.23}$$

Condition IIIc. $U \in \mathfrak{A}_{\text{IIIc}}$ iff for some onstants $K < \infty$ any $\gamma > 0$ and for the same V, Λ, t, $\bar{\sigma}^1$, $\bar{\sigma}^2$ as in IIIb

$$\operatorname{Var}(Q^U_{V,\Lambda}(\cdot \mid \bar{\sigma}^1), Q^U_{V,\Lambda}(\cdot \mid \bar{\sigma}^2)) \leqslant K \exp[-\gamma \operatorname{dist}(t, \Lambda)] \tag{2.24}$$

Condition IIId. $U \in \mathfrak{A}_{\text{IIId}}$ iff for some $K < \infty$ and $\gamma > 0$, for the same V, Λ, t, $\bar{\sigma}^1$, $\bar{\sigma}^2$ as in IIIb, and for all $\sigma_\Lambda \in \Omega_\Lambda$

$$\left| \frac{Q^U_{V,\Lambda}(\{\sigma_\Lambda\} \mid \bar{\sigma}^1)}{Q^U_{V,\Lambda}(\{\sigma_\Lambda\} \mid \bar{\sigma}^2)} - 1 \right| \leqslant K \exp[-\gamma \operatorname{dist}(t, \Lambda)] \tag{2.25}$$

Condition IVa. $U \in \mathfrak{A}_{\text{IVa}}$ iff for all $V \subset \mathbb{Z}^\nu$ and b.c. $\bar{\sigma} \in \Omega$ the following expansion holds:

$$\ln Z_V(U \mid \bar{\sigma}) = \sum_{t \in V} g(t, V, \bar{\sigma}) \tag{2.26}$$

where the function $g(\cdot, \cdot, \cdot)$ of the triples $t \in \mathbb{Z}^\nu$, $V \subset \mathbb{Z}^\nu$, $\bar{\sigma} \in \Omega$ has the following regularity properties:

(i) $g(\cdot, \cdot, \cdot)$ is translation-invariant, i.e., for all $s \in \mathbb{Z}^\nu$, $g(t, V, \bar{\sigma}) = g(t + s, V + s, \bar{\sigma}_{+s})$.

(ii) There exist constants $C < \infty$ and $c > 0$ such that for all V_1, V_2, $t \in V_1 \cap V_2$, $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$

$$|g(t, V_1, \bar{\sigma}^1) - g(t, V_2, \bar{\sigma}^2)| \leqslant C \exp[-c \operatorname{dist}(t, \Delta)] \tag{2.27}$$

where

$$\begin{aligned} \Delta &= \Delta(\bar{\sigma}^1, V_1, \bar{\sigma}^2, V_2) \\ &= [(V_1 \cup V_2) \backslash (V_1 \cap V_2)] \cup \{t \in (V_1 \cup V_2)^c : \bar{\sigma}^1_t \neq \bar{\sigma}^2_t\} \end{aligned}$$

Condition IVb. $U \in \mathfrak{A}_{\text{IVb}}$ iff for all $V \subset \mathbb{Z}^\nu$ and $\bar{\sigma} \in \Omega$ the following expansion holds:

$$\ln Z_V(U \mid \bar{\sigma}) = \sum_{t \in \partial V} \hat{g}(t, V, \bar{\sigma}) + \hat{g}\, |V| \tag{2.28}$$

where

$$\tilde{\partial} V = \{t \in V: \text{dist}(t, V^c) = 1\}$$

the function $\hat{g}(\cdot, \cdot, \cdot)$ of the triples $t \in \mathbb{Z}^\nu$, $V \subset \mathbb{Z}^\nu$, $\bar{\sigma} \in \Omega$, $t \in \tilde{\partial} V$ has the same properties (i) and (ii) as the function $g(\cdot, \cdot, \cdot)$ from IVa, and $\hat{g} = \hat{g}(U)$ is some constant.

In Ref. 1 we have effectively described several classes of CA potentials; namely, the cases of high temperature, large chemical potential, low temperature with unique ground state, and one-dimensional systems.

The main body of the above conditions are discussed in detail in Ref. 1, where references to earlier work can be found. In what follows we comment on the conditons of group IV, which are not to be found in Ref. 1, as well as on condition Ia, which now has a far more general form.

Remark 3. The conditions of group IV arise naturally in statistical mechanics problems. The expansion (2.26) was obtained in Ref. 5 for the Ising ferromagnet at low and high temperatures with constant b.c. (see also Ref. 6). In the form (2.28) it was used in Ref. 7 (again with constant b.c.). The additional requirement that the expansion holds for all b.c. is very essential. For example, in this generality it does not hold for the low-temperature Ising model.

Remark 4. Applying IVb to the case of V a ν-dimensional parallelelepiped with the smallest side $l(V) \to \infty$ and with constant b.c. $\bar{\sigma} \in \Omega$, one obtains the asymptotic expansion

$$\ln Z_V(U \mid \bar{\sigma}) = \sum_{j=0}^{\nu} \hat{g}_j S_j + O(e^{-\alpha l})$$

where S_j is the total area of the j-dimensional faces of V (for $j = 0$ it is just 2^ν = the number of sites), while $\hat{g}_j$, $j = 0, \ldots, \nu$, $\alpha > 0$, are some U-dependent constants (compare with Ref. 5).

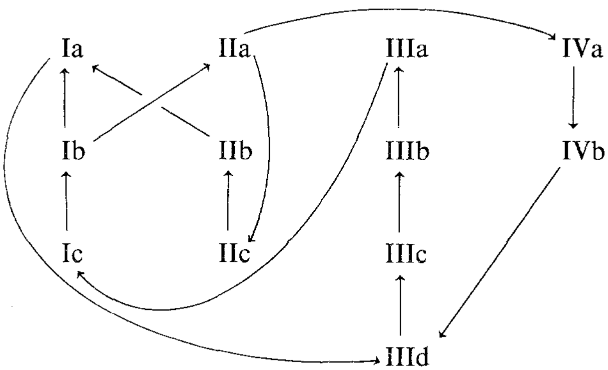
Remark 5. Comparing the conditions $\mathfrak{A}_\alpha$, it might appear strange that except for condition Ia, all contain several constant parameters on the r.h.s. of the corresponding bounds. How can this be, if these conditions can be derived from Ia, which has no parameters? The answer is that these parameters are obtained from easy *a priori* estimates on the partition function: the upper bound in complex space and the lower bound in real space (see Proposition 3.3).

Remark 6. The list of equivalent conditions can be further extended. For example, several conditions can be formulated only for translation-invariant perturbations (e.g., IIa). Some conditions can be formulated in stronger terms: for example, in Ia, instead of a nonzero partition function, one may ask for the bound

$$|\ln Z_V(\tilde{U} \mid \bar{\sigma})| \leqslant C\,|V|$$

to hold uniformly in V for all small, translation-invariant complex perturbations $\tilde{U} \in O_\varepsilon^T(U)$ (see Proposition 3.4). Again, it also can be formulated without the translation invariance condition.

The proof of Theorem 2.1 goes by showing that the following system of implications holds:



(2.29)

The arrow $X \to Y$ means that the condition Y holds for the main component of the interactions with the condition X.

It is easy to see that one can arrive at any vertex of (2.29) from any other, so it is enough to prove only these implications. In what follows we shall not discuss the implications already proven in Ref. 1 or immediate ones.[3] In the next section we present the proofs that are not found in Ref. 1.

3. TOWARD THE PROOF OF THEOREM 2.1

Proposition 3.1. $U \in \mathfrak{A}_{\mathrm{IIIa}} \Rightarrow U \in \mathfrak{A}_{\mathrm{Ic}}$.

Proof. It is easy to see that

$$\langle \varphi \rangle_{V,\bar{\sigma}}^{\tilde{U}} = \sum_{\sigma_W \in \Omega_W} \varphi(\sigma_W) \exp\Bigg\{ - \sum_{\substack{A:A \cap V \neq \varnothing, \\ A \cap V \subset W}} U_A(\sigma_W \cup \bar{\sigma}_{W^c}) \Bigg\} \times \frac{Z_{V \setminus W}(\tilde{U} \mid \sigma_W \cup \bar{\sigma}_{W^c})}{Z_V(\tilde{U} \mid \bar{\sigma})} \tag{3.1}$$

[3] Note that our condition Ib differs slightly from that of Ref. 1. Contrary to what was stated in Ref. 1, the implications from Ib being valid were not immediate.

Hence

$$|\langle\varphi\rangle_{V,\bar\sigma}^{\tilde U}|\leqslant \bar C\,\|\varphi\|\max_{\sigma_W\in\Omega_W}\left|\frac{Z_{V\setminus W}(\tilde U\,|\,\sigma_W\cup\bar\sigma_{W^c})}{Z_V(\tilde U\,|\,\bar\sigma)}\right| \tag{3.2}$$

for some $\bar C=\bar C(W, U, r, \nu, \varepsilon, |\mathscr{S}|)$.

We shall show that from Lemma 3.1 of Ref. 1, the conditions of which are satisfied for $U\in\mathfrak{A}_{\mathrm{IIIa}}$ and $\tilde U\in O_\varepsilon(U)$, it follows that

$$\left|\frac{Z_{V\setminus W}(\tilde U\,|\,\sigma_W\cup\bar\sigma_{W^c})}{Z_V(\tilde U\,|\,\bar\sigma)}\right|\leqslant C(|W|, U, r, \nu, \varepsilon, |\mathscr{S}|) \tag{3.3}$$

which together with (3.2) proves (2.9). Because $V\setminus W$ can be obtained from V by subsequently deleting points of $W\subset V$ one by one $|W|$ times, it is enough to consider the case $W=\{t\}$, which leaves us with the bound

$$\left|\frac{Z_{V\setminus\{t\}}(\tilde U\,|\,\sigma_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})}{Z_V(\tilde U\,|\,\bar\sigma)}\right|\leqslant C(U, r, \nu, \varepsilon, |S|) \tag{3.4}$$

But

$$Z_V(\tilde U\,|\,\bar\sigma)=\sum_{\tau_t\in S} Z_{V\setminus\{t\}}(\tilde U\,|\,\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})\times\exp\left\{-\sum_{A:A\cap V=\{t\}}\tilde U_A(\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})\right\} \tag{3.5}$$

and U is real; hence, it is enough to show that

$$\begin{aligned}&\left\{Z_{V\setminus\{t\}}(\tilde U\,|\,\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})\exp\left[-\sum_{A:A\cap V=\{t\}}\tilde U_A(\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})\right]\right\}\\&\quad\times[Z_{V\setminus\{t\}}(\tilde U\,|\,\sigma_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})]^{-1}\\&=\frac{Z_{V\setminus\{t\}}(U|\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})}{Z_{V\setminus\{t\}}(U|\sigma_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})}\exp\left[-\sum_{A:A\cap V=\{t\}}U_A(\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})\right](1+C_1\vartheta_1)\end{aligned} \tag{3.6}$$

where $C_1=C_1(U, r, \nu, \varepsilon)\to 0$ for $\varepsilon\to 0$, $\vartheta_1=\vartheta_1(\tau_t,\bar\sigma)$, $|\vartheta_1|\leqslant 1$.

But it follows from Lemma 3.1 of Ref. 1, statement II, that the ratio

$$\frac{Z_{V\setminus\{t\}}(\tilde U\,|\,\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})}{Z_{V\setminus\{t\}}(\tilde U\,|\,\sigma_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})}=(1+C_2\vartheta_2)\frac{Z_{V\setminus\{t\}}(U|\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})}{Z_{V\setminus\{t\}}(U|\sigma_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus\{t\}})}$$

while for $\tilde U\in O_\varepsilon(U)$

$$\exp\left[-\sum_{A:A\cap V=\{t\}}\tilde U_A(\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus t})\right]=\exp\left[-\sum_{A:A\cap V=\{t\}}U_A(\tau_t\cup\bar\sigma_{\mathbb{Z}^\nu\setminus t})\right](1+C_3\vartheta_3) \tag{3.7}$$

where C_2, C_3, ϑ_2, and ϑ_3 have the same properties as C_1 and ϑ_1; hence, (3.6) follows.

Proposition 3.2. $U \in \mathfrak{A}_{\mathrm{Ic}} \Rightarrow U \in \mathfrak{A}_{\mathrm{Ib}}$.

Proof. Let $\tilde{U}_1$ and $\tilde{U}_2$ be two perturbations of U. Consider the sequence of perturbations $\hat{U}^i \in O_\varepsilon(U)$ of U, $i = 1,..., k$, defined as follows:

(i) $\hat{U}^1|_{V \cup \partial V} = \tilde{U}_1|_{V \cup \partial V}, \qquad \hat{U}^k|_{V \cup \partial V} = \tilde{U}_2|_{V \cup \partial V}$ (3.8)

(ii) For some $A_1,..., A_{k-1} \in \Delta(\mathscr{A})$, the statement $(\hat{U}^{i+1} - \hat{U}^i)_A \not\equiv 0$ implies $A = A_i$, $i = 1,..., k-1$.

(iii) k is the smallest number satisfying (i) and (ii).

In this case clearly

$$k \leqslant C_1 \, |(V \cup \partial V) \cap \operatorname{supp}(\tilde{U}_1 - \tilde{U}_2)| \tag{3.9}$$

where $C_1 = C_1(v, r)$. To prove (2.8), it is enough to combine (3.9) with the bound

$$|\ln[Z_V(\hat{U}^i \,|\, \bar{\sigma})/Z_V(\hat{U}^{i+1} \,|\, \bar{\sigma})]| \leqslant C_2 \tag{3.10}$$

where $C_2 = C_2(\varepsilon, U, r, v)$. To show (3.10), let us consider the perturbation $\hat{U}^t = t\hat{U}^i + (1-t)\hat{U}^{i+1}$ and the function

$$F(t) = \ln[Z_V(\hat{U}^t \,|\, \bar{\sigma})/Z_V(\hat{U}^{i+1} \,|\, \bar{\sigma})], \qquad 0 \leqslant t \leqslant 1 \tag{3.11}$$

One has

$$F(0) = 0, \qquad F(1) = \ln[Z_V(\hat{U}^i \,|\, \bar{\sigma})/Z_V(\hat{U}^{i+1} \,|\, \bar{\sigma})] \tag{3.12}$$

$$F'(t) = \langle (\hat{U}^{i+1} - \hat{U}^i)_{A_i} \rangle_{V,\bar{\sigma}}^{\hat{U}^t} \tag{3.13}$$

The interaction $\hat{U}^t \in O_\varepsilon(U)$ for all $t \in [0, 1]$; hence, from (2.9) and the bound $|A_i| \leqslant r^v$ we have

$$|F'(t) \leqslant \varepsilon \tilde{C}(r^v, U, r, v, \varepsilon, |\mathscr{S}|) \tag{3.14}$$

which, together with (3.12), proves (3.10) and (2.8).

Proposition 3.3. $U \in M(\mathfrak{A}_{\mathrm{Ia}}) \Rightarrow U \in \mathfrak{A}_{\mathrm{IIId}}$.

First we prove the following:

Lemma 3.1. Suppose the function $\varphi(z)$ is analytic in the disk $\{|z| < 1 + \delta\}$, $\delta > 0$, with $|\varphi(z)| \leqslant C_1$ for $|z| \leqslant 1$ and $\varphi(0)$ is real, $\varphi(0) > \alpha > 0$. Then, for some $C_2 = C_2(C_1, \alpha)$, $E = E(C_1, \alpha)$, $0 < E < 1$,

$$\begin{aligned} \varphi(z) &\neq 0 \qquad \text{for} \quad |z| \leqslant E \\ |\ln \varphi(z)| &\leqslant C_2 \qquad \text{for} \quad |z| \leqslant E \end{aligned} \tag{3.15}$$

(Here we choose the branch of the log in such a way that the log is real.)

Proof of the Lemma. From the Cauchy formula it follows that

$$|\varphi'(z)| \leqslant C_1(1-|z|)^{-1} \qquad \text{for} \quad |z|<1 \tag{3.16}$$

Integrating along the segment $[0, z]$, we have

$$|\varphi(z)| > \alpha - C_1 \ln \frac{1}{1-|z|} > \frac{\alpha}{2} \tag{3.17}$$

if

$$|z| < E = 1 - \exp(-\alpha/2C_1) \tag{3.18}$$

For this region also

$$\ln \varphi(z) = \int_0^z \frac{\varphi'(u)}{\varphi(u)}\, du + \ln \varphi(0) \tag{3.19}$$

where the integral is taken along the segment $[0, z]$. Together with (3.16) and (3.17), this implies that for $|z| < E$

$$|\ln \varphi(z)| \leqslant 1 + |\ln \varphi(0)|$$

and (3.15) follows with

$$C_2 = 1 + \max\{|\ln \alpha|, |\ln C_1|\} \tag{3.20}$$

which finishes the proof of the lemma.

To prove Proposition 3.3, define for $\tilde{U} \in O_\varepsilon^T(U)$ the interaction $\hat{U}(z) = U + z(\tilde{U} - U)$ and note that for any $V \subset \mathbb{Z}^\nu$ and $\bar{\sigma} \in \Omega$ the function

$$\varphi(z) = Z_V(\hat{U}(z) \mid \bar{\sigma})^{1/|V|} \tag{3.21}$$

is analytic for $|z| \leqslant 1$. (This is the only place where we need the partition function to be nonzero.) Let

$$\bar{u} = \sup_{A, \sigma_A \subset \Omega_A} |U_A(\sigma_A)| \tag{3.22}$$

Then for some $\kappa = \kappa(\nu, r)$

$$|Z_V(\tilde{U} \mid \bar{\sigma})^{1/|V|}| \leqslant \exp[\kappa(\bar{u} + \varepsilon + \ln |\mathscr{S}|)]$$

provided $\tilde{U} \in O_\varepsilon^T(U)$. This bound holds, in particular, for $\tilde{U} = \hat{U}(z)$, $0 \leqslant |z| \leqslant 1$. Because U is real, we have also

$$Z_V(U \mid \bar{\sigma})^{1/|V|} \geqslant \exp(-\kappa \bar{u})$$

Hence, we can apply to $\varphi(z)$ Lemma 3.1, which gives the following result: for any $V \subset \mathbb{Z}^\nu$, $\bar{\sigma} \in \Omega$, and $\tilde{U} \in O_{\varepsilon'}^T(U)$,

$$|\ln Z_V(\tilde{U} \mid \bar{\sigma})| \leqslant [1 + \kappa(\bar{u} + \varepsilon + \ln |\mathscr{S}|)]\, |V| \tag{3.23}$$

provided

$$\varepsilon' = (1 - \exp\{-\exp[-\kappa(2\bar{u} + \varepsilon + \ln |\mathscr{S}|)]\})\varepsilon \tag{3.24}$$

The rest of the proof follows identically the same lines as that of Proposition 4.3 of Ref. 1.

Proposition 3.4. $U \in M(\mathfrak{A}_{\mathrm{IIa}}) \Rightarrow U \in \mathfrak{A}_{\mathrm{IVa}}$.

Proof. Let us join U with the zero interaction U^0 by the smooth path $U^t \in M(\mathfrak{A}_{\mathrm{IIa}})$ in the manner discussed in Section 4 of Ref. 1, $U^1 = U$. Then

$$\ln Z_V(U \mid \bar{\sigma}) = \int_0^1 [\ln Z_V(U^t \mid \bar{\sigma})]'_t \, dt + |V| \ln |\mathscr{S}| \tag{3.25}$$

But

$$[\ln Z_V(U^t \mid \bar{\sigma})]'_t = \left\langle - \sum_{A : A \cap V \neq \varnothing} (U^t_A)'_t (\sigma_V \cup \bar{\sigma}_{V^c}) \right\rangle_{V, \bar{\sigma}}^{U^t}$$

where all the function $[U^t_A(G_A)]'_t$ are uniformly bounded in t. Let

$$g(s, V, \bar{\sigma}) = -\int_0^1 dt \left\langle \sum_{A : s \in A} \frac{1}{|A|} (U^t_A)'_t (\sigma_V \cup \bar{\sigma}_{V^c}) \right\rangle_{V, \bar{\sigma}}^{U^t} + \ln |\mathscr{S}| \tag{3.26}$$

Clearly, the representation (2.26) holds. Now, from (4.30) of Ref. 1 the bound (2.27) follows immediately.

Proposition 3.5. $U \in \mathfrak{A}_{\mathrm{IVb}} \Rightarrow U \in \mathfrak{A}_{\mathrm{IIId}}$.

Proof. One easily checks that

$$\begin{aligned}
&\ln \frac{Q^U_V((\sigma_A) \mid \bar{\sigma}^1)}{Q^U_V((\sigma_A) \mid \bar{\sigma}^2)} \\
&\quad = \ln \frac{Z_{V \setminus A}(U \mid \bar{\sigma}^1_{A^c} \cup \sigma_A)/Z_V(U \mid \bar{\sigma}^1)}{Z_{V \setminus A}(U \mid \bar{\sigma}^2_{A^c} \cup \sigma_A)/Z_V(U \mid \bar{\sigma}^2)} \\
&\quad = \sum_{s \in \partial(V \setminus A) - \partial(V)} |\hat{g}(s, V \setminus A, \bar{\sigma}^1) - \hat{g}(s, V \setminus A, \bar{\sigma}^2)| \\
&\qquad + \sum_{\substack{s \in \partial(V) \cap (V \setminus A): \\ \mathrm{dist}(s,t) \leqslant 1/2 \, \mathrm{dist}(A,t)}} [|\hat{g}(s, V \setminus A, \bar{\sigma}^1) - \hat{g}(s, V, \bar{\sigma}^1)| \\
&\qquad + |\hat{g}(s, V \setminus A, \bar{\sigma}^2) - \hat{g}(s, V, \bar{\sigma}^2)|] \\
&\qquad + \sum_{\substack{s \in \partial(V) \cap (V \setminus A): \\ \mathrm{dist}(s,t) > 1/2 \, \mathrm{dist}(A,t)}} [|\hat{g}(s, V \setminus A, \bar{\sigma}^1) - \hat{g}(s, V \setminus A, \bar{\sigma}^2)| \\
&\qquad + |\hat{g}(s, V, \bar{\sigma}^1) - \hat{g}(s, V, \bar{\sigma}^2)|] \\
&\qquad + \sum_{s \in \partial(V) \cap A} |\hat{g}(s, V, \bar{\sigma}^1) - \hat{g}(s, V, \bar{\sigma}^2)|
\end{aligned}$$

Because $\bar{\sigma}_s^1 = \bar{\sigma}_s^2$ for $s \neq t$, one has by IVb a bound of the type $C' \exp[-\frac{1}{2}c \operatorname{dist}(t, \Lambda)]$ for each term of each sum, with $C' = C'(C, c, \nu)$; hence IIId follows.

Proposition 3.6. $U \in \mathfrak{A}_{\mathrm{IVa}} \Rightarrow U \in \mathfrak{A}_{\mathrm{IVb}}$.

Proof. The limit

$$\hat{g} = \lim_{\operatorname{dist}(t, V^c) \to \infty} g(t, V, \bar{\sigma})$$

exists and does not depend on $\bar{\sigma}$, because of (2.26). For any $s \in V$ denote by $\mathscr{D}(s)$ the subset of points t of ∂V such that

$$\operatorname{dist}(s, t) = \min_{\tau \in \partial V} \operatorname{dist}(s, \tau)$$

and define

$$\hat{g}(t, V, \bar{\sigma}) = \sum_{s \in V: t \in \mathscr{D}(s)} |\mathscr{D}(s)|^{-1} [g(s, V, \bar{\sigma}) - \hat{g}]$$

The relation (2.28) evidently holds. A bound of the type (2.27) for $\hat{g}(\cdot, \cdot, \cdot)$ can be easily obtained from that for $g(\cdot, \cdot, \cdot)$ (with the same c but with other C; compare with preceding Proposition 3.5 or with Ref. 7, Sect. 3).

Remark. We use the opportunity to fill a small gap in the proof of Proposition 4.3 in Ref. 1. Instead of the discussion after the bound (4.28), one has to reason as follows. From (4.28) of Ref. 1 and the estimates on $|\tilde{V}|$ and $|\partial \tilde{V}|$ after it, one infers that

$$\left| \frac{Q_{V,\Lambda}^U(\{\sigma_\Lambda\} \mid \bar{\sigma}^1)}{Q_{V,\Lambda}^U(\{\sigma_\Lambda\} \mid \bar{\sigma}^2)} - 1 \right| \leqslant K |\Lambda|^2 \exp[-\gamma \operatorname{dist}(t, \Lambda)] \tag{3.27}$$

for all $\gamma < \gamma(U)$ and some $K = K(\gamma) < \infty$. Defining $\Lambda' = \Lambda'(\Lambda)$ to be

$$\Lambda' = \{s \in V: \operatorname{dist}(t, \Lambda) \leqslant |t - s| < \operatorname{dist}(t, \Lambda) + r\} \tag{3.28}$$

and using an analog of (4.27) of Ref. 1, one has

$$\max_{\sigma_\Lambda} \left| \frac{Q_{V,\Lambda}^U(\{\sigma_\Lambda\} \mid \bar{\sigma}^1)}{Q_{V,\Lambda}^U(\{\sigma_\Lambda\} \mid \bar{\sigma}^2)} - 1 \right| \leqslant \max_{\sigma_{\Lambda'}} \left| \frac{Q_{V,\Lambda'}^U(\{\sigma_{\Lambda'}\} \mid \bar{\sigma}^1)}{Q_{V,\Lambda'}^U(\{\sigma_{\Lambda'}\} \mid \bar{\sigma}^2)} - 1 \right| \tag{3.29}$$

Hence, to prove (2.25), one can use (3.27) with Λ' instead of Λ. But $|\Lambda'| \leqslant C(\operatorname{dist}(t, \Lambda))^\nu$ with some $C = C(r, \nu)$, hence (2.25) follows from (3.27).

4. THE CONSTRUCTIVE CA CONDITIONS

As was already mentioned, each of the 12 conditions has its constructive counterpart. We shall present only three—the most characteristic. (In what follows, the parameters ν and r are fixed, and usually omitted.)

Constructive Condition Ia. $U \in \mathfrak{A}_{\mathrm{Ia}}^{\mathrm{constr}}(d; \varepsilon, C)$ if $\|U\| < C$ and for all $V \subset \mathbb{Z}^\nu$ with diam $V \leqslant d$ the function $Z_V(\tilde{U} \mid \bar{\sigma})$ is nonzero inside $O_\varepsilon^T(U)$ for all b.c. $\bar{\sigma}$.

Constructive Condition IIb. $U \in \mathfrak{A}_{\mathrm{IIb}}^{\mathrm{constr}}(d; \varphi, C)$ with function φ, satisfying (2.17) if the bound (2.16) holds for all volumes V with diam $V \leqslant d$.

Constructive Condition IIIc. $U \in \mathfrak{A}_{\mathrm{IIIc}}^{\mathrm{constr}}(d; K, \gamma)$ if the bound (2.24) holds for all volumes V with diam $V \leqslant d$.

In the same manner the other conditions of Section 2 are made constructive. It is convenient to have a unique notation g_α for the set of constants of the condition $\mathfrak{A}_\alpha^{\mathrm{constr}}(d; \cdot)$. Thus, g_α denotes

ε, C	for	$\alpha = \mathrm{Ia, Ib, IIa}$
$\varepsilon, \tilde{C}$	for	$\alpha = \mathrm{Ic}$
$\varphi(\cdot), C$	for	$\alpha = \mathrm{IIb}$
δ, C	for	$\alpha = \mathrm{IIc}$
δ, ρ	for	$\alpha = \mathrm{IIIa}$
$\varphi(\cdot)$	for	$\alpha = \mathrm{IIIb}$
γ, K	for	$\alpha = \mathrm{IIIc, IIId}$
c, C	for	$\alpha = \mathrm{IVa, IVb}$

The corresponding condition will henceforth be denoted by $\mathfrak{A}_\alpha^{\mathrm{constr}}(d; g_\alpha)$ or simply $\mathfrak{A}_\alpha^{\mathrm{constr}}(d)$.

Theorem 4.1. For each $\alpha = \mathrm{Ia}, \ldots, \mathrm{IVb}$ there exists a function $d_0 = d_0^\alpha(g_\alpha)$ such that $M(\mathfrak{A}_\alpha^{\mathrm{constr}}(d_0; g_\alpha))$ coinsides with the set of CA interactions.

It is possible to give explicit expressions for the functions d_0. For example, for $\alpha = \mathrm{IIIc}$,

$$d_0^{\mathrm{IIIc}}(K, \gamma) = \min \left\{ d\colon [(K+1)(d+2r+1)^\nu]^{2\nu+3} \exp(-\gamma d/3\nu) < \frac{1}{2\nu} \right\} \quad (4.1)$$

where min is taken over all integers $d > 6\nu(r+1)$ that are multiples of 3ν. However, the bound (4.1) does not have to be taken too seriously. In deciding between a more accurate bound and a simpler proof, we chose the latter, and so the bound (4.1) is greatly excessive.

Theorem 4.1 follows immediately from the following two statements.

Proposition 4.1. The set $M(\mathfrak{A}_{\mathrm{IIIc}}^{\mathrm{constr}}(d_0))$, with d_0 given by (4.1), consists of CA interactions.

Proposition 4.2. Suppose that for $\alpha_1, \alpha_2 = \mathrm{Ia},..., \mathrm{IVb}$ there is an arrow, $\mathfrak{A}_{\alpha_1} \to \mathfrak{A}_{\alpha_2}$, in (2.29). Then there exists a function $g_{\alpha_2}(g_{\alpha_1})$ such that for all d, g_{α_1}

$$U \in M(\mathfrak{A}_{\alpha_1}^{\mathrm{constr}}(d, g_{\alpha_1})) \Rightarrow U \in \mathfrak{A}_{\alpha_2}^{\mathrm{constr}}(d, g_{\alpha_2}(g_{\alpha_1}))$$

The proof of Proposition 4.1 is given below in Sections 5 and 6. The proof of Proposition 4.2 follows quite easily using Ref. 1 and Section 3 of this paper. One has only to check that when deriving some property of the field in a volume V, one has used only the information confined to this V. Going over these proofs, one can also obtain explicit formulas for the functions g_{α_2} (g_{α_1}). If the function $d_0^{\alpha_2}(g_{\alpha_2})$ of Theorem 4.1 is already known, one can take

$$d_0^{\alpha_1}(g_{\alpha_1}) = d_0^{\alpha_2}(g_{\alpha_2}(g_{\alpha_1})) \tag{4.2}$$

From (4.1) and (4.2) it is possible to determine all the functions d_0^{α} for the conditions $\mathfrak{A}_{\alpha}^{\mathrm{constr}}$.

5. THE STRATEGY OF THE PROOF OF PROPOSITON 4.1, OR: HOW TO CLEAN A BIG TABLE WITH A SMALL DUSTER?

Let the b.c. $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ differ only at $t \in \partial V$. We want to show that the conditional distributions $Q_{V,\Lambda}^U(\cdot, \mid \bar{\sigma}_1)$ and $Q_{V,\Lambda}^U(\cdot \mid \bar{\sigma}^2)$ are exponentially close [as $\mathrm{dist}(t, \Lambda) \to \infty$] for all $\Lambda \subset V$, all $V \subset \mathbb{Z}^\nu$, provided it is known only for those volumes V whose diameter is less than or equal to d_0. To this end, we use the surgery method introduced in Ref. 8, which has since been intensively used (see, e.g., Refs. 3 and 9). Its main ideas are the following.

Let P^1, P^2 be two probability distributions on Ω_V, $V \subset \mathbb{Z}^\nu$. The distribution P on $\Omega_V \times \Omega_V$ is called a joint distribution for P_1, P_2 if

$$\begin{aligned} \sum_{\sigma_V^1 \in \Omega_V} P(\sigma_V^1, \sigma_V^2) &= P^2(\sigma_V^2) \\ \sum_{\sigma_V^2 \in \Omega_V} P(\sigma_V^1, \sigma_V^2) &= P^1(\sigma_V^1) \end{aligned} \tag{5.1}$$

We say that the pair P^1, P^2 is f-close, where f is a real valued function on V, iff there is a joint distribution P for P^1, P^2 such that

$$\langle \rho_s \rangle_P = \sum_{\sigma^1_V, \sigma^2_V \in \Omega_V} \rho_s(\sigma^1_V, \sigma^2_V)\, P(\sigma^1_V, \sigma^2_V) \leqslant f(s), \qquad s \in V \tag{5.2}$$

where for all $V \subset \mathbb{Z}^\nu$, $s \in V$, $\sigma^1_V, \sigma^2_V \in \Omega_V$

$$\rho_s(\sigma^1_V, \sigma^2_V) = \begin{cases} 1, & \sigma^1_s \neq \sigma^2_s \\ 0, & \sigma^1_s = \sigma^2_s \end{cases} \tag{5.3}$$

The following simple statement explains the connection between f-closeness and variation distance.

Lemma 5.1. Let $\Lambda \subset V$ and

$$P^i_\Lambda(\sigma_\Lambda) = \sum_{\sigma_{V\setminus\Lambda} \in \Omega_{V\setminus\Lambda}} P^i(\sigma_\Lambda \cup \sigma_{V\setminus\Lambda}) \tag{5.4}$$

be the restriction of P^i onto Λ.

(i) If P^1, P^2 are f-close, then

$$\operatorname{Var}(P^1_\Lambda, P^2_\Lambda) \leqslant \sum_{s \in \Lambda} f(s)$$

(ii) Any pair P^1, P^2 is f-close with

$$f(s) = \begin{cases} \operatorname{Var}(P^1_\Lambda, P^2_\Lambda), & s \in \Lambda \\ 1, & s \in V\setminus\Lambda \end{cases} \tag{5.5}$$

Proof. (i) Let χ_S, $S \subset \Omega_\Lambda$, be the indicator of the set $\{\sigma_V \in \Omega_V\colon \sigma_\Lambda \in S\}$. One can easily follow the following sequence of inequalities:

$$\begin{aligned}
\operatorname{Var}(P^1_\Lambda, P^2_\Lambda) &= \max_{S \subset \Omega_\Lambda} |P^1_\Lambda(S) - P^2_\Lambda(S)| \\
&= \max_{S \subset \Omega_\Lambda} \left| \sum_{\sigma_V \in \Omega_V} \chi_S(\sigma_V)\, P^1(\sigma_V) - \sum_{\sigma_V \in \Omega_V} \chi_S(\sigma_V)\, P^2(\sigma_V) \right| \\
&= \max_{S \subset \Omega_\Lambda} \left| \sum_{\sigma^1_V, \sigma^2_V \in \Omega_V} [\chi_S(\sigma^1_V) - \chi_S(\sigma^2_V)]\, P(\sigma^1_V, \sigma^2_V) \right| \\
&\leqslant \max_{S \subset \Omega_\Lambda} \sum_{\sigma^1_V, \sigma^2_V \in \Omega_\Lambda} |\chi_S(\sigma^1_V) - \chi_S(\sigma^2_V)|\, P(\sigma^1_V, \sigma^2_V) \\
&\leqslant \sum_{\sigma^1_V, \sigma^2_V \in \Omega_V} \left[\sum_{s \in \Lambda} \rho_s(\sigma^1_V, \sigma^2_V) \right] P(\sigma^1_V, \sigma^2_V) \\
&= \sum_{s \in \Lambda} \langle \rho_s \rangle_P \leqslant \sum_{s \in \Lambda} f(s)
\end{aligned}$$

which proves (i).

(ii) Let

$$Q_\Lambda(\sigma_\Lambda) = \min(P^1_\Lambda(\sigma_\Lambda), P^2_\Lambda(\sigma_\Lambda))$$
$$\hat{P}^i_\Lambda(\sigma_\Lambda) = P^i_\Lambda(\sigma_\Lambda) - Q_\Lambda(\sigma_\Lambda)$$

Then

$$\sum_{\sigma_\Lambda \in \Omega_\Lambda} Q_\Lambda(\sigma_\Lambda) = 1 - \mathrm{Var}(P^1_\Lambda, P^2_\Lambda) \tag{5.6}$$

Consider now the joint distribution for the pair P^1_Λ, P^2_Λ, given by the formula

$$P_\Lambda(\sigma^1_\Lambda, \sigma^2_\Lambda) = \begin{cases} Q_\Lambda(\sigma_\Lambda), & \sigma^1_\Lambda = \sigma^2_\Lambda = \sigma_\Lambda \\ \hat{P}^1_\Lambda(\sigma^1_\Lambda)\, \hat{P}^2_\Lambda(\sigma^2_\Lambda)/\mathrm{Var}(P^1_\Lambda, P^2_\Lambda), & \sigma^1_\Lambda \neq \sigma^2_\Lambda \end{cases}$$

Let $P^i(\sigma_{V\setminus\Lambda} \mid \sigma_\Lambda)$ be the conditional distributions on $V\setminus\Lambda$ subject to the condition σ_Λ on Λ, corresponding to P^i. We can define the joint distribution P for P^1, P^2 by

$$P(\sigma^1_V, \sigma^2_V) = P_\Lambda(\sigma^1_\Lambda, \sigma^2_\Lambda)\, P^1(\sigma^1_{V\setminus\Lambda} \mid \sigma^1_\Lambda)\, P^2(\sigma^2_{V\setminus\Lambda} \mid \sigma^2_\Lambda)$$

One can easily estimate now that

$$\langle \rho_s \rangle_P \leqslant \begin{cases} \mathrm{Var}(P^1_\Lambda, P^2_\Lambda) & \text{for} \quad s \in \Lambda \\ 1 & \text{for} \quad s \in V\setminus\Lambda \end{cases}$$

which proves (ii).

It is possible now to reformulate the main Proposition 4.1 in terms of closeness. We define a set $\hat{\mathfrak{A}}(d; K, \gamma)$ of interactions to consist of all potentials U such that for all $V \subset \mathbb{Z}^\nu$ with diam $V \leqslant d$, all $\Lambda \subset V$, $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ the pair $Q^U_V(\cdot \mid \bar{\sigma}^1)$, $Q^U_V(\cdot \mid \bar{\sigma}^2)$ is $\varphi^\Lambda_{\bar{\sigma}^1,\bar{\sigma}^2}(\cdot)$ close, with

$$\varphi^\Lambda_{\bar{\sigma}^1,\bar{\sigma}^2}(s) = \begin{cases} K \sum\limits_{t \in \partial V} \exp[-\gamma\, \mathrm{dist}(t, \Lambda)]\, \rho_t(\bar{\sigma}^1, \bar{\sigma}^2) & \text{for} \quad s \in \Lambda \\ 1 & \text{for} \quad s \in V\setminus\Lambda \end{cases} \tag{5.7}$$

Proposition 5.2. $U \in \mathfrak{A}^{\mathrm{constr}}_{\mathrm{IIIc}}(d; K, \gamma) \Rightarrow U \in \hat{\mathfrak{A}}(d; K, \gamma)$.

Proof. Let $\{t_1, ..., t_n\}$ be the sequence of all points of $t \in \partial V$ where the b.c. $\bar{\sigma}^1$, $\bar{\sigma}^2$ differ. Consider then the sequence of b.c. $\bar{\bar{\sigma}}^i$, $i = 0, ..., n$, given by

$$\bar{\bar{\sigma}}^i_t = \begin{cases} \bar{\sigma}^1_t & \text{for} \quad t \in \{t_1, ..., t_i\} \\ \bar{\sigma}^2_t & \text{for other} \quad t - s \end{cases}$$

From the triangle inequality it follows that

$$\mathrm{Var}(Q^U_{V,A}(\cdot\,|\,\bar\sigma^1), Q^U_{V,A}(\cdot\,|\,\bar\sigma^2)) \leqslant \sum_{i=1}^{n} \mathrm{Var}(Q^U_{V,A}(\cdot\,|\,\bar{\bar\sigma}^{i-1}), Q^U_{V,A}(\cdot\,|\,\bar{\bar\sigma}^{i}))$$

where we have used that $Q^U_{V,A}(\cdot\,|\,\bar\sigma^1) = Q^U_{V,A}(\cdot\,|\,\bar{\bar\sigma}^n)$. Because the b.c. $\bar{\bar\sigma}^{i-1}$, $\bar{\bar\sigma}^{i}$ differ exactly in one point t_i, it follows from (2.24) that

$$\mathrm{Var}(Q^U_{V,A}(\cdot\,|\,\bar{\bar\sigma}^{i-1}), Q^U_{V,A}(\cdot\,|\,\bar{\bar\sigma}^{i})) \leqslant K \exp[-\gamma\, \mathrm{dist}(t_i, A)]$$

The bound (5.7) follows now from statement (ii) of Lemma 5.1.

The above proposition enables us to use for the proof of Proposition 4.1 the condition $U \in \hat{\mathfrak{A}}(d_0; L, \gamma)$ instead of $U \in \mathfrak{A}^{\mathrm{constr}}_{\mathrm{IIIc}}(d_0; L, \gamma)$.

Section 6 deals with the proof of the following (nontrivial) statement, which in fact contains all the difficult points of our problem.

Proposition 5.3. If $U \in \hat{\mathfrak{A}}(d_0; L, \gamma)$, where d_0 is given by (4.1), then for some constants L' and γ' and for all volumes $V \subset \mathbb{Z}^\nu$, all $t_0 \in \partial V$, and all pairs $\bar\sigma^1, \bar\sigma^2 \in \Omega$ of boundary conditions that differ only at t_0, the pair $Q^U_A(\cdot\,|\,\bar\sigma^1)$, $Q^U_A(\cdot\,|\,\bar\sigma^2)$ is f-close with

$$f(s) = L' \exp(-\gamma' \, |s - t_0|) \tag{5.8}$$

Proposition 4.1 clearly follows from Proposition 5.3. Indeed, from (5.8) and part (i) of Lemma 5.1 it follows that

$$\begin{aligned} &\mathrm{Var}(Q^U_{V,A}(\cdot\,|\,\bar\sigma^1), Q^U_{V,A}(\cdot\,|\,\bar\sigma^2)) \\ &\quad\leqslant \sum_{s \in A} L' \exp[-\gamma' \, \mathrm{dist}(t_0, s)] \\ &\quad\leqslant \sum_{u \in \mathbb{Z}^\nu, |u| \geqslant \mathrm{dist}(t_0, A)} L' \exp(-\gamma' \, |u|) \leqslant L'' \exp[-\gamma'' \, \mathrm{dist}(t_0, A)] \quad (1.9) \end{aligned}$$

where $0 < \gamma'' < \gamma'$ and the constant $L'' = L''(L', \gamma', \gamma'')$. Hence, $U \in \mathfrak{A}_{\mathrm{IIIc}}$. From Proposition 5.2 it follows now that $M(\mathfrak{A}^{\mathrm{constr}}_{\mathrm{IIIc}}(d_0)) \subset \mathfrak{A}_{\mathrm{IIIc}}$, so the statement of Proposition 4.1 follows from the main Theorem 2.1.

The proof of Proposition 5.3 is more involved, so we begin with the main ideas.

For any volume W with diam $W \leqslant d_0$, any $A \subset W$, let us fix a joint distribution $P^A_W(\sigma^1_W, \sigma^2_W\,|\,\bar\sigma^1, \bar\sigma^2)$ for the pair $Q^U_W(\sigma^1_W\,|\,\bar\sigma^1)$, $Q^U_W(\sigma^2_W\,|\,\bar\sigma^2)$, corresponding to the pair $\bar\sigma^1, \bar\sigma^2 \in \Omega$ of b.c., such that

$$\langle \rho_s \rangle_{P^A_W(\cdot,\cdot\,|\,\bar\sigma^1, \bar\sigma^2)} \leqslant \varphi^A_{\bar\sigma^1, \bar\sigma^2}(s), \qquad s \in W$$

where $\varphi^{\Lambda}_{\bar\sigma^1,\bar\sigma^2}$ is defined in (5.7) if $\bar\sigma^1 \neq \bar\sigma^2$, while for the pair of coincident b.c. $\bar\sigma^1 = \bar\sigma^2 = \bar\sigma$ the joint distribution $P^{\alpha}_W(\cdot, \cdot \mid \bar\sigma, \bar\sigma)$ lies on the diagonal $\sigma^1_W = \sigma^2_W$. Such a system exists in the case $U \in \mathfrak{A}(d_0; L. \gamma)$. We call the volumes W with diam $W \leqslant d_0$ patterns, and the distributions P^{Λ}_W pattern distributions.

The following is the main surgery procedure applied to the joint distribution in arbitrarily large volume. It consists of a surgery in a fixed pattern, and it results in f-closeness with "better" [≡ smaller (but not everywhere)] f.

Lemma 5.4. (Elementary Surgery Lemma).

Let W be a pattern, $\Lambda \subset W$, and $P^{\Lambda}_W(\cdot, \cdot \mid \bar\sigma^1, \bar\sigma^2)$ be a pattern distribution. Suppose that $W \subset V$, $\bar\sigma^1, \bar\sigma^2 \in \Omega$, and $\Pi_V(\cdot, \cdot \mid \bar\sigma^1, \bar\sigma^2)$ is some joint distribution for $Q^U_V(\cdot \mid \bar\sigma^1)$, $Q^U_V(\cdot \mid \bar\sigma^2)$. Let

$$\tilde\Pi_V(\sigma^1_V, \sigma^2_V \mid \bar\sigma^1, \bar\sigma^2) = P^{\Lambda}_W(\sigma^1_W, \sigma^2_W \mid \sigma^1_{V\setminus W} \cup \bar\sigma^1_{(V\setminus W)^c}, \sigma^2_{V\setminus W} \cup \bar\sigma^2_{(V\setminus W)^c}) \times \Pi_{V,V\setminus W}(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W} \mid \bar\sigma^1, \bar\sigma^2) \tag{5.10}$$

where

$$\Pi_{V,V\setminus W}(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W} \mid \bar\sigma^1, \bar\sigma^2) = \sum_{\tau^1_W, \tau^2_W \in \Omega_W} \Pi_V(\tau^1_W \cup \sigma^1_{V\setminus W}, \tau^2_W \cup \sigma^2_W \mid \bar\sigma^1, \bar\sigma^2)$$

is the restriction of Π_V on $V\setminus W$. Then $\tilde\Pi_V$ is also a joint distribution for $Q^U_V(\cdot \mid \bar\sigma^1)$, $Q^U_V(\cdot \mid \bar\sigma^2)$. Moreover, if

$$\langle \rho_u \rangle_{\Pi_V} \leqslant f(u), \qquad u \in V \tag{5.11}$$

then

$$\langle \rho_u \rangle_{\tilde\Pi_V} \leqslant \begin{cases} L \sum_{s \in \partial W} f(s) \exp[-\gamma \operatorname{dist}(s, \Lambda)], & u \in \Lambda \\ \sum_{s \in \partial W} f(s), & u \in W\setminus\Lambda \\ f(u), & u \in V\setminus W \end{cases} \tag{5.12}$$

if we define for $s \in V^c$

$$f(s) = \begin{cases} 1, & \bar\sigma^1_s \neq \bar\sigma^2_s \\ 0, & \bar\sigma^1_s = \bar\sigma^2_s \end{cases}$$

and where $\langle \rho_u \rangle_{\Pi_V}$ and $\langle \rho_u \rangle_{\tilde\Pi_V}$ are shorthand for

$$\langle \rho_u \rangle_{\Pi_V(\cdot,\cdot \mid \bar\sigma^1, \bar\sigma^2)} \qquad \text{and} \qquad \langle \rho_u \rangle_{\tilde\Pi_V(\cdot,\cdot \mid \bar\sigma^1, \bar\sigma^2)}$$

Proof. That $\tilde\Pi_V$ is a joint distribution follows from the fact that P^{Λ}_W

and Π_V are, and from the definition of Gibbs field. To prove (5.12), one has to observe that for $u \in \Lambda$ if follows from (5.9), (5.7), and (5.11) that

$$\begin{aligned}
\langle \rho_u \rangle_{\tilde{\Pi}_V} = & \sum_{\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W}} \Pi_{V,V\setminus W}(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W} \mid \bar{\sigma}^1, \bar{\sigma}^2) \\
& \times \langle \rho_u \rangle_{P^{\Lambda}_W(\cdot,\cdot \mid \sigma^1_{V\setminus W} \cup \bar{\sigma}^1_{(V\setminus W)^c}, \sigma^2_{V\setminus W} \cup \bar{\sigma}^2_{(V\setminus W)^c})} \\
\leqslant & L \Bigg\{ \sum_{\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W}} \Pi_{V,V\setminus W}(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W} \mid \bar{\sigma}^1, \bar{\sigma}^2) \\
& \times \sum_{s \in \partial W \cap V} \exp[-\gamma \operatorname{dist}(s, \Lambda)] \, \rho_s(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W}) \\
& + \sum_{s \in \partial W \cap V^c} \exp[-\gamma \operatorname{dist}(s, \Lambda)] \, \rho_s(\bar{\sigma}^1, \bar{\sigma}^2) \Bigg\} \\
= & L \Bigg\{ \sum_{s \in \partial W \cap V} \exp[-\gamma \operatorname{dist}(s, \Lambda)] \langle \rho_s \rangle_{\Pi_V} \\
& + \sum_{s \in \partial W \cap V^c} \exp[-\gamma \operatorname{dist}(s, \Lambda)] \, f(s) \Bigg\} \\
\leqslant & L \sum_{s \in \partial W} f(s) \exp[-\gamma \operatorname{dist}(s, \Lambda)]
\end{aligned}$$

which proves the first line in (5.12). Going on to the case $u \in W \setminus \Lambda$ and using the fact that pattern distribution lies on the diagonal for coincident b.c., one has for $\bar{\sigma}^1_{\partial W \cap V^c} = \bar{\sigma}^2_{\partial W \cap V^c}$ [otherwise the second line in (5.12) is trivial]

$$\begin{aligned}
\langle \rho_u \rangle_{\tilde{\Pi}_V} \leqslant & \sum_{\substack{\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W} \in \Omega_{V\setminus W} \\ \sigma^1_{V \cap \partial W} \neq \sigma^2_{V \cap \partial W}}} \Pi_{V,V\setminus W}(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W} \mid \bar{\sigma}^1, \bar{\sigma}^2) \\
\leqslant & \sum_{\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W}} \left[\sum_{s \in V \cap \partial W} \rho_s(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W}) \right] \Pi_{V,V\setminus W}(\sigma^1_{V\setminus W}, \sigma^2_{V\setminus W} \mid \bar{\sigma}^1, \bar{\sigma}^2) \\
= & \sum_{s \in \partial W \cap V} \langle \rho_s \rangle_{\Pi_V} \leqslant \sum_{s \in W} f(s)
\end{aligned}$$

Finally, it follows from the Definition (5.10) that for $u \in V \setminus W$

$$\langle \rho_u \rangle_{\tilde{\Pi}_V} = \langle \rho_u \rangle_{\Pi_V} \leqslant f(u)$$

which finishes the proof of Lemma 5.4.

The following is the idea of the proof of Lemma 5.3, which is transformed into the proof in the next section.

From Lemma 5.4 one knows that the result of an elementary surgery

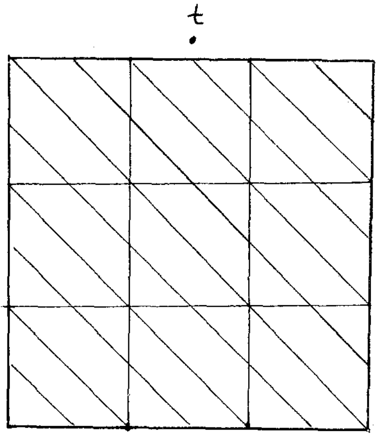


Fig. 1. The initial dirt density.

of the measure Π_V is some new measure $\tilde{\Pi}_V$, which is somehow "better" in $\Lambda \subset W$, is the same outside W, but may be "worse" in $W \setminus \Lambda$. To form a clear intuitive picture about what is going on, use Aizenman's description of the process, who proposed to view the values of the function $f(\cdot)$ as the amount of "dirt" at each point. Then the surgery in the pattern $W \subset V$ can be viewed as rubbing on W with a duster. This cleaning goes on, however, in a masculine fashion, i.e., not very carefully, with the result that the amount of dirt decreases only in the center of W, with the dirt being removed to the boundary of W. Moreover, it can freely be that the total amount of dirt even increases! In any case, the center of W becomes

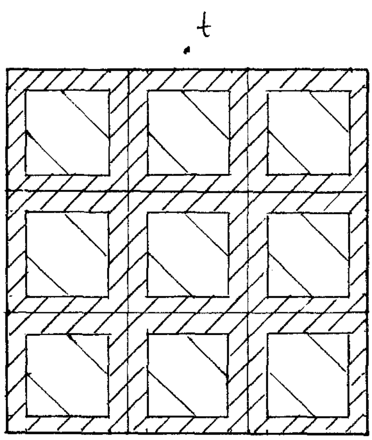


Fig. 2. Dirt distribution after first cleaning.

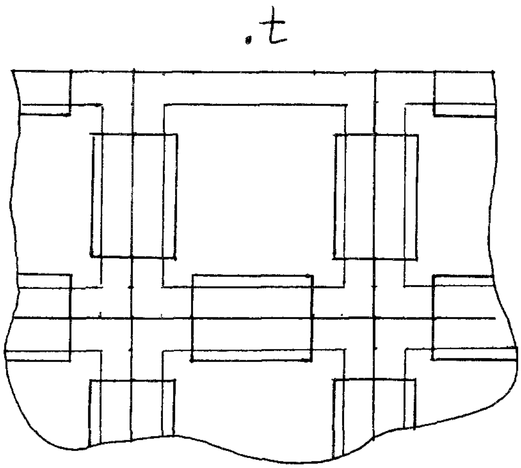


Fig. 3. The positions of the second cleanings.

cleaner, while the dirt lies near the boundary, its amount being proportional to that in the vicinity of ∂W. So, to clean the whole table one can proceed as indicated in Figs. 1–8, where the two-dimensional case is considered, the shading representing the amount of dirt. First we cover all the table $\hat{V}_1$ with two-dimensional patterns (Fig. 1) and perform cleaning in each of them. (To be more precise, one has to arrange the patterns in such a way that their mutual distances are greater than r. But in this section we shall ignore these details.) After this the dirt is shifted to the boundaries of the patterns (Fig. 2). Next, one has to cover the pieces of these boundaries

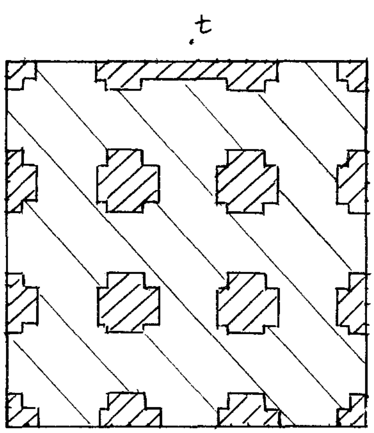


Fig. 4. The result of the second cleaning.

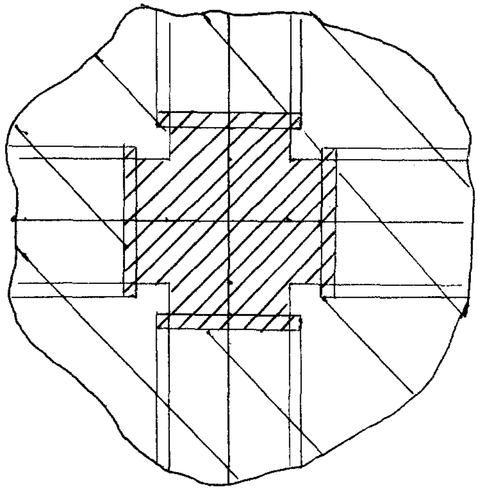


Fig. 5. Magnified piece of Fig. 4.

by "almost one-dimensional" patterns, which have to be disjoint (see Fig. 3). The only piece not covered is that in the vicinity of the point t, where the outer dirt is situated. The result of the second cleaning is shown in Fig. 4, while Fig. 5 contains a magnified piece of Fig. 4, which shows the dirt distribution in the vicinity of the common certer of four two-dimensional patterns. The additional dirt, created after second cleaning, is easily seen. Finally, Fig. 6 shows the position of "almost zero-dimensional" patterns. After cleaning inside them the general situation is considerably imporved, and the table is cleaneer everywhere except in the neigborhood of the point t. One may summarize as follows: after each cleaning the "dimension" of the exceptionally dirty parts is reduced by one, while its

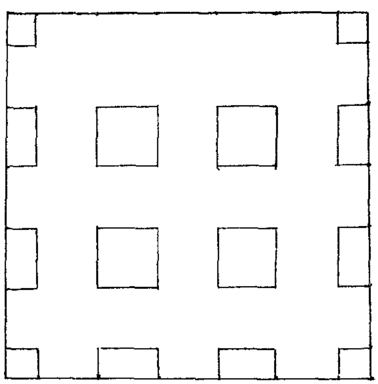


Fig. 6. The positions of the third cleaning.

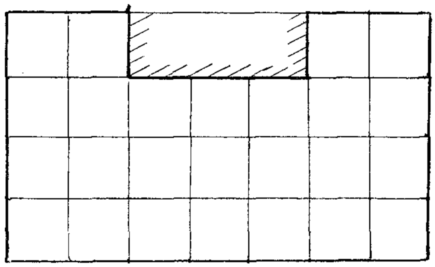


Figure 7

thickness grows, until at the last step they disappear completely except in the vicinity of the point t.

One would like to iterate the above scheme. But it will not give any result in the patterns adjacent to the point t. So one has to repeat the procedure outside these patterns, i.e., to consider "the table" $\hat{V}_2 = \hat{V}_1 \setminus \bigcup W^*$ (see Fig. 7). The situation on the table $\hat{V}_2$ is of the same type as that for $\hat{V}_1$, the boundary of $\hat{V}_2$ being clean outside the dashed region. What is important here is that the height of dirt on the boundary of $\hat{V}_2$ is in constant times smaller than that for $\hat{V}_1$, provided the patterns W are big enough.

Iterating this procedure, i.e., applying it on the nth step to the volume $\hat{V}_n$ (see Fig. 8), one gets the desired result.

6. THE PROOF OF LEMMA 5.3

We are left with the proof of Lemma 5.3, and we shall follow the plan outlined in the previous section. We begin with a purely analytic reformulation of the result sought, so in this section there will be no probability.

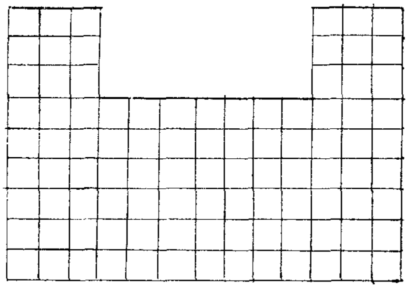


Figure 8

Let V, $t \in \partial V$ be fixed. Denote by $\mathcal{F} = \mathcal{F}(V, t, d_0, K, \gamma)$ the class of nonnegative functions $f = \{f(s), s \in V\}$ with the properties:

(i) The identity function is in $\mathcal{F}$.

(ii) If $f \in \mathcal{F}$ and

$$\tilde{f}(s) \geqslant \min(f(s), 1), \qquad s \in V \tag{6.1}$$

then $\tilde{f} \in \mathcal{F}$.

(iii) If $f \in \mathcal{F}$ and $\Lambda \subset W \subset V$ are volumes with diam $W \leqslant d_0$, then also $\hat{f} = S_{W,\Lambda} f \in \mathcal{F}$, where

$$\hat{f}(u) = \begin{cases} K \sum\limits_{s \in \partial W \cap V} f(s) \exp[-\gamma \operatorname{dist}(s, \Lambda)] \\ \qquad + \chi_W \exp[-\gamma \operatorname{dist}(t, \Lambda)], & u \in \Lambda \\ \sum\limits_{s \in \partial W \cap V} f(s) + \chi_W, & u \in W \backslash \Lambda \\ f(u), & u \in V \backslash W \end{cases} \tag{6.2}$$

Here $\chi_W = 1$ for $t \in \partial W$, $\chi_W = 0$ otherwise.

The connection between this definition and the preceding section is given by the following:

Lemma 6.1. Let $U \in \mathfrak{A}(d_0; K, \gamma)$, where d_0 is given by (4.1). Suppose that b.c. $\bar{\sigma}^1, \bar{\sigma}^2$ are given, with $\bar{\sigma}_s^1 = \bar{\sigma}_s^2$ for $s \neq t$. Then the pair $Q_V^U(\cdot \mid \bar{\sigma}^1)$, $Q_V^U(\cdot \mid \bar{\sigma}^2)$ of conditional Gibbs distributions is f-close for any $f \in \mathcal{F}(V, t, d_0, K, \gamma)$.

Proof. To show that $Q_V^U(\cdot \mid \bar{\sigma}^1)$, $Q_V^U(\cdot \mid \bar{\sigma}^2)$ are 1-close, it is enough to apply statement (ii) of Lemma 5.1 with $\Lambda = \varnothing$. To see that f-closeness implies $\tilde{f}$-closeness [see (6.1)], one uses the definition (5.2) and the fact that $\langle \rho_s \rangle_P$ is always less than 1. The last statement, that f-closeness implies $\hat{f}$-closeness [see (6.2)] is a reformulation of the Elementary Surgery Lemma 5.4.

In what follows, we shall also call surgery the transformation $f \to \hat{f} = S_{W,\Lambda} f$. Instead of Proposition 5.3 we shall prove the following.

Proposition 6.2. For any finite V the function

$$f(s) = K' \exp[-\gamma' \operatorname{dist}(t, s)] \tag{6.3}$$

is in $\mathcal{F}$, where

$$K' = 2B, \qquad \gamma' = [2(d_0 + 2r + 1)]^{-1} \tag{6.4}$$

with d_0 as in (4.1) and with

$$B=[(K+1)(d_0+2r+1)^{\nu}]^{\nu+1} \tag{6.5}$$

One sees easily that Proposition 6.2 and Lemma 6.1 together imply Proposition 5.3.

The following is the formalization of the sequence of surgeries described at the end of Section 5.

Lemma 6.3. Let $\hat{V}\subset V$, and suppose that $\varphi\in\mathscr{F}$, with

$$\varphi(s)\leqslant c\leqslant 1, \qquad s\in\hat{V} \tag{6.6}$$

Then $\hat{\varphi}\in\mathscr{F}$, where

$$\hat{\varphi}(s)=\begin{cases}\varphi(s), & s\in V\setminus\hat{V}\\ Bc, & s\in\hat{V},\ \mathrm{dist}(s,(V\setminus\hat{V})\cup t)<D\\ bc, & s\in\hat{V},\ \mathrm{dist}(s,(V\setminus\hat{V})\cup t)\geqslant D\end{cases} \tag{6.7}$$

where B is as in (6.5), d_0 as in (4.1),

$$b=\nu B^2K(d_0+2r+1)^{\nu}\exp(-\gamma\, d_0/3\nu) \tag{6.8}$$

$$D=2(d_0+2r+1) \tag{6.9}$$

The proof of Lemma 6.3 is somewhat lengthy and we postpone it, while we now explain that Proposition 6.2, and hence Theorem 4, follow from it. Let $T_{\hat{V}}$ be the operator taking φ into $\hat{\varphi}$, defined by (6.7). Consider the sequence of volumes $V_1=V$, $V_2,\dots,V_n$, where

$$V_{i+1}=\{s\in V_i\colon \mathrm{dist}(s,(V\setminus V_i)\cup t)\geqslant D\} \tag{6.10}$$

and where n is the smallest value of i with V_{i+1} empty. Applying the oeprators T_{V_i} to the function $\varphi_1(s)\equiv 1\in\mathscr{F}$ [see (i) of the definition of the class $\mathscr{F}$], we have a sequence

$$\varphi_{i+1}=T_{V_i}\varphi_i, \qquad i=1,2,\dots,n-1 \tag{6.11}$$

which is in $\mathscr{F}$ because of Lemma 6.3. From (6.7) it follows that

$$\varphi_n(s)\leqslant Bb^{i-1} \qquad \text{for} \quad t\in V_i\setminus V_{i+1} \tag{6.12}$$

We choose the number d_0 in such a way that it ensures that b is less than 1/2, hence the function φ_n decays exponentially.

Using (6.12) and the definitions (6.4)–(6.5), one is easily convinced that the function f of (6.3) obeys

$$f(s) \geqslant \tilde{\varphi}_n(s) = \min\{\varphi_n(s), 1\}$$

Hence Proposition 6.2 follows from the definition of $\mathscr{F}$.

Proof of Lemma 6.3. This will consist of consecutive applications to the function φ, obeying (6.6), of the operators $S_{W,\Lambda}$ defined in (6.2), along the lines of Section 5. Define the operator

$$S_{\hat{V}} = \prod_{k=0}^{\nu} \left(\prod_{\mathscr{W}_k} S_{W_{k,i}, \Lambda_{k,i}} \right) \tag{6.13}$$

where $\mathscr{W}_k = \mathscr{W}_k(V, \hat{V}, t) = \{W_{k,i}\}$ are the pattern families to be defined, $\Lambda_{k,i} \subset W_{k,i}$. We shall show that

$$S_{\hat{V}} \varphi \leqslant \hat{\varphi} \tag{6.14}$$

with $\hat{\varphi}$ from (6.7), which, according to the definition of the class $\mathscr{F}$, is enough for Lemma 6.3 to hold.

The definitions of $W_{k,i} \subset \hat{V}$ and $\Lambda_{k,i} \subset W_{k,i}$ were outlined in Section 5. We begin by first stating their geometric properties that are crucial for our proof, their presentation is postponed until the end of this section. There are five such properties.

P1. For all k, $W_{k,i} \in \mathscr{W}_k$,

$$W_{k,i} \subset \hat{V}, \qquad \operatorname{diam} W_{k,i} \leqslant d_0,$$
$$\partial W_{k,i} \cap [(V \backslash \hat{V}) \cup t] = \varnothing$$

where d_0 is as in (4.1).

P2. For all $k = 0,..., \nu$, $i_1 \neq i_2$,

$$\operatorname{dist}(W_{k,i_1}, W_{k,i_2}) > r \tag{6.15}$$

Let us define the volumes $\hat{V}_k$, $k = 0,..., \nu + 1$, by the recursion

$$\hat{V}_{\nu+1} = \varnothing$$
$$\hat{V}_k = \left[\hat{V}_{k+1} \Big\backslash \bigcup_i (W_{k,i} \backslash \Lambda_{k,i}) \right] \cup \left(\bigcup_i \Lambda_{k,i} \right) \tag{6.16}$$

P3. The following inclusion holds:

$$\{s \in \hat{V} \colon \operatorname{dist}(s, (V \backslash \hat{V}) \cup t) > D\} \subset \hat{V}_0 \tag{6.17}$$

where D is defined in (6.9).

P4. For all $k=1,..., \nu$ (except $k=0!$), all i,

$$\operatorname{dist}(\Lambda_{k,i}, W_{k,i}^c \backslash \hat{V}_{k+1}) > d_0/3\nu \tag{6.18}$$

P5. For all i

$$\partial W_{0,i} \subset \hat{V}_1, \qquad \Lambda_{0,i} = W_{0,i}$$

We now shall demonstrate (6.14) provided the system of patterns with P1–P5 exists. Note that in general the operator $S_{\hat{V}}$ is not well-defined by (6.13), because the operators $S_{W,\Lambda}$ generally do not commute. But in our case the situation is different: according to P2, the operators $S_{W_{k,i},\Lambda_{k,i}}$ and $S_{W_{k,j},\Lambda_{k,j}}$ do commut for the same values of k, while their order for different k is as prescribed by (6.13): one begins with ν-dimensional surgeries, then follow with $(\nu-1)$-dimensional ones, and so on.

Let us introduce the intermediate operators

$$S_{\hat{V}}^{(j)} = \prod_{k=j}^{\nu} \prod_{\mathscr{W}_k} Q_{W_{k,i},\Lambda_{k,i}}, \qquad j=0,..., \nu, \quad S_{\hat{V}}^{(0)} = S_{\hat{V}} \tag{6.19}$$

We begin by obtaining a rough estimate on $\hat{\varphi}^{(j)} = S_{\hat{V}}^{(j)} \varphi$. Namely, let us show that for all $j=0,..., \nu$

$$\hat{\varphi}^{(j)} \leqslant Bc \tag{6.20}$$

with B from (6.5). The bound (6.20) is rough because it cannot be improved only in the vicinity of $(V \cap \partial \hat{V}) \cup t$ [see the second line in (6.7)].

To see (6.20), we first estimate from above the number of points in $\partial W_{k,i}$ (for any k, i) by $(d_0+R)^\nu$, where we put $R=2r+1$ in order to simplify the notations. Hence, by definition (6.2) and P1 it follows that if $f(s) \leqslant a$, $s \in \hat{V}$, then

$$(S_{W_{k,i},\Lambda_{k,i}} f)(s) \leqslant \begin{cases} K(d_0+R)^\nu, & s \in W_{k,i} \\ a, & s \in \hat{V} \backslash W_{k,i} \end{cases} \tag{6.21}$$

But any point $s \in \hat{V}$ is at most in one $W_{k,i}$ for any given k (see P2). Hence, by (6.19), we have for all $j=0,..., \nu$

$$|\hat{\varphi}^{(j)}(s)| \leqslant B_1^{\nu+1-j} c \leqslant Bc \tag{6.22}$$

where

$$B_1 = K(d_0+R)^\nu \tag{6.23}$$

In the same way the estimate (6.25) can be checked: let i, j be fixed, $W = W_{j,i}$, $\Lambda = \Lambda_{j,i}$, and suppose that

$$\hat{\varphi}^{(j+1)}(s) \leqslant \alpha$$

(where $\hat{\varphi}^{(v+1)} \equiv \varphi$) provided s is in

$$n(\Lambda) = \{s \in \partial W: \text{dist}(s, \Lambda) \leqslant d_0/3v\} \tag{6.24}$$

Then

$$\hat{\varphi}^{(j)}(s) \leqslant \alpha B_1 + Bcb_1 \qquad \text{for} \quad s \in \Lambda \tag{6.25}$$

with

$$b_1 = K(d_0 + R)^v \exp(-\gamma(d_0/3v)) \tag{6.26}$$

Our definitions of surgeries imply also that

$$\hat{\varphi}^{(j)}(s) = \hat{\varphi}^{(j+1)}(s) \qquad \text{as long as} \quad s \in \hat{V} \Big\backslash \bigcup_i W_{j,i} \tag{6.27}$$

Now everything is ready for the inductive estimates on $\hat{\varphi}^{(j)}$ to obtain. By (6.25) and (6.19), the function $\hat{\varphi}^{(v)}$ satisfies, in addition to (6.20), also the bound

$$\hat{\varphi}^{(v)}(s) \leqslant Bcb_1 \tag{6.28}$$

for s in the set

$$\hat{V}^{(v)} = \bigcup_i \Lambda_{v,i} \tag{6.29}$$

[see (6.16)]. Only the second term of (6.25) contributes to (6.28); the first one vanishes because the set $n(\Lambda_{v,i}) = \varnothing$ by (6.18), and so one can set α to be zero.

The bound (6.28) on $\hat{V}^{(v)}$ when incorporated into (6.25), and using (6.18), results in

$$\hat{\varphi}^{(v-1)}(s) \leqslant Bcb_1 B_1 + Bcb_1 \tag{6.30}$$

for $s \in \hat{V}^{(v-1)}$, provided $v - 1 > 0$. Here we also have used (6.27). Iterating and using the bound (6.25) to estimate the function $\hat{\varphi}^{(j)}$, $j > 0$, with α given by the rhs of the bound on the function $\hat{\varphi}^{(j+1)}$ on the set $\hat{V}^{(j+1)}$ we arrive at

$$\hat{\varphi}^{(j)}(s) \leqslant Bcb_1(1 + B_1 + \cdots + B_1^{v-j}) \tag{6.31}$$

for $s \in \hat{V}^{(j)}$.

At a last step one has to invoke condition P5 to bound the function $\hat{\varphi}^{(0)}$. Because $|\partial W_{0,i}| \leqslant (d_0 + R)^v$, we have, using the already proven bound on $\hat{\varphi}^{(1)}$, that

$$\hat{\varphi}^{(0)}(s) \leqslant Bcb_1(B_1 + \cdots + B_1^v) \leqslant bc \tag{6.32}$$

for $s \in \hat{V}^{(0)}$, which, together with (6.17), proves (6.14), because $B_1^{\nu} + B_1^{\nu-1} + \cdots + B_1 \leqslant \nu B$ and

$$S_{\hat{V}}\varphi = \hat{\varphi}^{(0)} \tag{6.33}$$

We are left with the presentation of the patterns with properties P1–P5.

Let L be the cube in $\mathbb{R}^{\nu}$, centered at the origin, of the size $d_0 + R$, oriented according to the coordinate axes. Let $\mathscr{L}$ be a covering of $\mathbb{R}^{\nu}$ generated by shifts of L by the vectors from the sublattice $(d_0 + R)\mathbb{Z}^{\nu}$. First, we present families $\mathscr{M}_k$ of parallelepipeds, or boxes, in $\mathbb{R}^{\nu}$, $k = 0,..., \nu$. The family $\mathscr{M}_k$ is formed by "almost k-dimensional" boxes. Their centers are those of (all) k-dimensional faces of cubes forming the covering $\mathscr{L}$. If x is such a center and $\Gamma(x)$ the corresponding k-dimensional face, then all the sides of the box $\Pi^k(x) \in \mathscr{M}_k$ centered at x are parallel to the axes; these that are parallel to the face $\Gamma(x)$ have the length

$$d_0 - (\nu - k)(R + 2d_0/3\nu) \tag{6.34}$$

while the rest have length equal to

$$(\nu - k)(R + 2d_0/3\nu) \tag{6.35}$$

From (6.34) and (6.35) and since $d_0 > 6\nu(r+1) > 3\nu R$ [see (4.1)]

$$\operatorname{dist}(\Pi^k(x') \cap \mathbb{Z}^{\nu}, \Pi^k(x'') \cap \mathbb{Z}^{\nu}) \geqslant R > r \tag{6.36}$$

for all k, $x' \neq x''$.

Let x be the center of a k-dimensional face $\Gamma(x)$ of a cube from $\mathscr{L}$. Denote by $\mathscr{K}^{(l)}(x)$, $l > k$, the set of all centers of all l-dimensional faces of the cubes from $\mathscr{L}$, which faces contain the face $\Gamma(x)$. By definition, the nonempty intersection $\Pi^k(x) \cap \hat{V}$ belongs to $\mathscr{W}_k$ iff

$$\begin{aligned} &\{[\Pi^k(x) \cap \mathbb{Z}^{\nu}] \cup \partial_r[\Pi^k(x) \cap \mathbb{Z}^{\nu}]\} \cap [(V \backslash \hat{V}) \cup t] = \varnothing \\ &\{[\Pi^l(y) \cap \mathbb{Z}^{\nu}] \cup \partial_r[\Pi^l(y) \cap \mathbb{Z}^{\nu}]\} \cap [(V \backslash \hat{V}) \cup t] = \varnothing \end{aligned} \tag{6.37}$$

for all $y \in \bigcup_{l > k} \mathscr{K}^{(l)}(x)$.

To define the subvolumes $\Lambda_{k,i} \subset W_{k,i}$, let $\Pi^k(x) \in \mathscr{M}_k$ and let $\Gamma(x)$ be the corresponding k-dimensional face. The intersection $\Pi^k(x) \cap \Gamma(x)$ is a "real" k-dimensional box. Let it be denoted by $\tilde{\Pi}^k(x)$, and let $\partial\tilde{\Pi}^k(x)$ be its boundary. We define the "$(k-1)$-dimensional" boundary $\partial^{(k-1)}\Pi^k(x)$ of "k-dimensional" box $\Pi^k(x)$, $k > 0$, to be the set of the points of its true boundary that belong to those faces of this box $\Pi^k(x)$ that contain the points of $\partial\tilde{\Pi}^k(x)$. For example, for $k=v$, $\partial^{(v-1)}=\partial$. We define the subvolume $\Lambda_{k,i}$ for the pattern $W_{k,i}=\Pi^k(x)\cap\hat{V}$, $k>0$, $i=i(x)$, by

$$\Lambda_{k,i}=\{s\in W_{k,i}\colon \mathrm{dist}(s,(\partial^{(k-1)}\Pi^k(x)\cap\mathbb{Z}^v))\geqslant d_0/3v\} \tag{6.38}$$

(keeping in mind that $\Lambda_{0,i}=W_{0,i}$).

The property P1 follows by definition of the patterns [see (6.37)], while P2 follows from (6.36).

To prove P3 one has to show first by induction in k (beginning with $k=v$ in decreasing order, to $k=0$) that the subsets $\tilde{V}_k\subset\hat{V}$ of points s defined by:

(i) the distances between s and all $(k-1)$-dimensional faces Γ of cubes of $\mathscr{L}$ are greater than $(v-k)(R/2+d_0/3v)$

(ii) there exist $j\geqslant k$, i such that $s\in\Lambda_{j,i}\subset W_{j,i}\in\mathscr{W}_j$, $j\leqslant v$

are in $\hat{V}_k$, $k=0,\ldots,v$. Note that, by definition, (-1)-dimensional face Γ_{-1} is empty, and $\mathrm{dist}(s,\varnothing)=+\infty$. It remains to observe that the subsets $\tilde{\tilde{V}}_k\subset\hat{V}$ of points s defined by (i) above and by

(ii′) $\mathrm{dist}(s,(V\setminus\hat{V})\cup t)>D$

belong to $\tilde{V}_k$, $k=0,\ldots,v$.

The conditions P4 and P5 follow straightforwardly.

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